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PG Semester 1

Paper – CC 2

Unit – 2

Topic – Jordan's Lemma

Jordan's ~~lemma~~ lemma

We shall now prove a very useful and important theorem which is usually known as Jordan's lemma.

Jordan's lemma:

Theorem:- If C_R is a semi circle with its centre at origin and radius R in the upper half plane and $f(z)$ satisfies the following conditions:

(i) It is analytic in the upper half plane except at a finite number of poles.

(ii) $f(z) \to 0$ uniformly as $|z| \to \infty$ for $0 \le \arg z \le \pi$ then

$$\lim_{R \to 0} \int_{C_R} e^{imz} f(z)\, dz = 0 \quad \text{--- (1)}$$

Where m is a positive number.

Proof:- From condition (ii) if R is sufficiently large, then there exists a positive number ϵ such that for all points z on C_R

$$|f(z)| < \epsilon \quad \text{--- (2)}$$

Further for all points on C_R, we have

$$z = Re^{i\theta}; \text{ so that } dz = iRe^{i\theta}d\theta$$

$$\therefore |e^{imz}| = |e^{im.R(\cos\theta + i\sin\theta)}|$$

$$= |e^{+im.R\cos\theta}||e^{-mR\sin\theta}|$$

$$= e^{-mR\sin\theta} \quad \text{——— (3)}$$

Hence
$$\left|\int_{C_R} e^{imz} f(z)\,dz\right| \leq \int_{C_R} |e^{imz}||f(z)||dz|$$

$$\leq \epsilon \int_0^{\pi} e^{-mR\sin\theta} |iRe^{i\theta}d\theta| = \epsilon \int_0^{\pi} e^{-mR\sin\theta}.R\,d\theta$$

$$= 2\epsilon R \int_0^{\pi/2} e^{-mR\sin\theta}\,d\theta \quad \text{——— (4)}$$

Now it can be shown by considering the sign of its derivative or otherwise that $\frac{\sin\theta}{\theta}$ decreases steadily from 1 to $\frac{2}{\pi}$ as θ increases from 0 to $\frac{\pi}{2}$ so that

$$\frac{\sin\theta}{\theta} \geq \frac{\sin \pi/2}{\frac{\pi}{2}} \text{ i.e. } \frac{\sin\theta}{\theta} \geq \frac{2}{\pi}$$

i.e. $\sin\theta \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$

Hence $e^{mR\sin\theta} \leq e^{-2mR\theta/\pi}$ — (5)

in view of (5), (4) takes the form

$$\left|\int_{C_R} e^{imz} f(z)\,dz\right| \leq 2\epsilon R \int_0^{\pi/2} e^{-2mR\theta/\pi} \cdot d\theta$$

$$= 2\epsilon R \frac{(1-e^{-mR})}{2mR/\pi} = \frac{\epsilon\pi}{m}\left(1-e^{-mR}\right)$$

$$< \frac{\epsilon\pi}{m} \quad \text{— (6)}$$

As $|z| = R \to \infty$, $|f(z)| < \epsilon \to 0$;

therefore inequality (6) leads

to $$\lim_{R\to\infty} \int_{C_R} e^{imz} f(z)\,dz = 0 \quad \text{— (7)}$$

By the use of Jordan's lemma we can evaluate the integrals of the form

$$\int_{-\infty}^{+\infty} f(x)\cos mx\,dx \text{ and } \int_{-\infty}^{+\infty} f(x)\sin mx\,dx \;(m>0) \quad \text{—} \; (8)$$

where the funⁿ $f(z)$ in the complex plane corresponding to given real funⁿ $f(x)$ satisfies the following conditions

(i) It is analytic in the upper half plane at a finite number of poles

(ii) It has no poles the real axis

(iii) $f(z) \to 0$ uniformly as $|z| \to \infty$ for $0 \leq \arg z \leq 2\pi$.

Accordingly $f(z)$ may be a rational funⁿ of the form $f(z) = \dfrac{p(z)}{q(z)}$ where $p(z)$ and $q(z)$ are polynomials with no factors in common and the degree of q is at least one unit higher than that of p; and according to condition (ii) $q(z)$ has no real roots.

To evaluate the integrals (8) let us integrate $f(z)e^{imz}$ counterclockwise around the boundary of contour C consisting of

(a) the semicircle C_R with centre at origin and sufficiently large radius R such that it includes all the poles of $f(z)$ in the upper half plane.

(b) the line segment of the real axis from $-R$ to $+R$. Then by Cauchy residue theorem, we have

$$\int_{-R}^{R} f(x) e^{imx} dx + \int_{C_R} f(z) e^{imz} dz = 2\pi i \sum R^+ \quad — (9)$$

where $\sum R^+$ denotes the sum of residues of $f(z) e^{imz}$ at its poles in the upper half plane.

Since $f(z) e^{imz}$ satisfies the conditions of Jordan's lemma, we have from (7)

$$\lim_{R \to \infty} \int_{C_R} e^{imz} f(z) dz = 0$$

Hence eqn (9) in the limit $R \to \infty$ yields

$$\int_{-\infty}^{+\infty} f(x)\, e^{imx}\, dx = 2\pi i \sum R^{+} \quad \text{—} (10)$$

Equating real and imaginary parts on both sides of (10) we can evaluate the integrals (8) as

$$\int_{-\infty}^{+\infty} f(x) \cos mx \, dx = -2\pi \sum \text{Im} \cdot \text{Res} \left[f(z) e^{imz} \right] \quad \text{—} (11a)$$

$$\int_{-\infty}^{+\infty} f(x) \sin mx \, dx = 2\pi \sum \text{Re} \cdot \text{Res} \cdot \left[f(z) e^{imz} \right] \quad \text{—} (11b)$$